

KRICHEVER CORRESPONDENCE FOR ALGEBRAIC SURFACES

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In 70's there was discovered a construction how to attach to some algebraic-geometric data an infinite-dimensional subspace in the space $k((z))$ of the Laurent power series. The construction was successfully used in the theory of integrable systems, particularly, for the KP and KdV equations [10, 19]. There were also found some applications to the moduli of algebraic curves [2, 3]. Now it is known as the Krichever correspondence or the Krichever map [2, 11, 1, 17, 4]. The original work by I. M. Krichever has also included commutative rings of differential operators as a third part of the correspondence.

The map we want to study here was first described in an explicit way by G. Segal and G. Wilson [19]. They have used an analytical version of the infinite dimensional Grassmanian introduced by M. Sato [18, 16]. In the sequel we consider a purely algebraic approach as developed in [11].

Let us just note that the core of the construction is an embedding of the affine coordinate ring on an algebraic curve into the field $k((z))$ corresponding to the power decompositions in a point at infinity (the details see below in section 2). In number theory this corresponds to an embedding of the ring of algebraic integers to the fields \mathbb{C} or \mathbb{R} . The latter one is well known starting from the XIX-th century. The idea introduced by Krichever was to insert the local parameter z . This trick looking so simple enormously extends the area of the correspondence. It allows to consider all algebraic curves simultaneously.

But there still remained a hard restriction by the case of curves, so by dimension 1. Recently, it was pointed out by the author [15] that there are some connections between the theory of the KP-equations and the theory of n -dimensional local fields [13], [6]. From this point of view it becomes clear that the Krichever construction should have a generalization to the case of higher dimensions. This

generalization is suggested in the paper for the case of algebraic surfaces (see theorem 3 in section 2). A further generalization to the case of arbitrary dimension was recently proposed by D. V. Osipov [12].

Let us also note that the construction of the restricted adelic complex in section 1 is of an independent interest, also in arithmetics. It has already appeared in a description of vector bundles on algebraic surfaces [14].

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1 Adelic Complexes

We first discuss the adelic complexes for the case of dimension 1. Concerning a definition of the adelic notions we refer to [6],[9]. We also note that the sign \prod denotes the adelic product.

Let C be an projective algebraic curve over a field k , P be a smooth point and η a general point on C . Let \mathcal{F} be a torsion free coherent sheaf on C .

Proposition 1 . *The following complexes are quasi-isomorphic:*

i) *adelic complex*

$$\mathcal{F}_\eta \oplus \prod_{x \in C} \hat{\mathcal{F}}_x \longrightarrow \prod_{x \in C} (\hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_x} K_x)$$

ii) *the complex*

$$W \oplus \hat{\mathcal{F}}_P \longrightarrow \hat{\mathcal{F}}_P \otimes_{\hat{\mathcal{O}}_P} K_P$$

where $W = \Gamma(C - P, \mathcal{F}) \subset \hat{\mathcal{F}}_\eta$.

PROOF will be done in two steps. First, the adelic complex contains a trivial exact subcomplex

$$\prod_{x \in U} \hat{\mathcal{F}}_x \longrightarrow \prod_{x \in U} \hat{\mathcal{F}}_x,$$

where $U = C - P$. The quotient-complex is equal to

$$\mathcal{F}_\eta \oplus \hat{\mathcal{F}}_P \longrightarrow \prod_{x \in U} (\hat{\mathcal{F}}_x \otimes K_x) / \hat{\mathcal{F}}_x \oplus \hat{\mathcal{F}}_P.$$

It has a surjective homomorphism to the exact complex

$$\mathcal{F}_\eta / W \longrightarrow \prod_{x \in U} (\hat{\mathcal{F}}_x \otimes K_x) / \hat{\mathcal{F}}_x.$$

The exactness of the complex is the strong approximation theorem for the curve C [5][ch.II, §3, corollary of prop. 9; ch. VII, §, prop. 2]. The kernel of this surjection will be the second complex from proposition.

Now we go to the case of dimension 2. Let X be a projective irreducible algebraic surface over a field k , $C \subset X$ be an irreducible projective curve, and $P \in C$ be a smooth point on both C and X . Let \mathcal{F} be a torsion free coherent sheaf on X .

DEFINITION 1. Let $x \in C$. We let

$$B_x(\mathcal{F}) = \bigcap_{D \neq C} ((\hat{\mathcal{F}}_x \otimes K_x) \cap (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D})),$$

where the intersection is done inside the group $\hat{\mathcal{F}}_x \otimes K_x$,

$$B_C(\mathcal{F}) = (\hat{\mathcal{F}}_C \otimes K_C) \cap \left(\bigcap_{x \neq P} B_x \right),$$

where the intersection is done inside $\hat{\mathcal{F}}_x \otimes K_{x,C}$,

$$A_C(\mathcal{F}) = B_C(\mathcal{F}) \cap \hat{\mathcal{F}}_C,$$

$$A(\mathcal{F}) = \hat{\mathcal{F}}_\eta \cap \left(\bigcap_{x \in X-C} \hat{\mathcal{F}}_x \right).$$

We will freely use the following shortcuts:

$$\begin{aligned} K\hat{\mathcal{F}}_x &= \hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_x} K_x, \\ K\hat{\mathcal{F}}_D &= \hat{\mathcal{F}}_D \otimes_{\hat{\mathcal{O}}_D} K_D, \\ \hat{\mathcal{F}}_{x,D} &= \hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_x} \mathcal{O}_{x,D}, \\ K\hat{\mathcal{F}}_{x,D} &= \hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_x} K_{x,D}. \end{aligned}$$

Next, we need two lemmas connecting the adelic complexes on X and C . They are the versions of the relative exact sequences, see [13], [6]. The curve C defines the following ideals:

$$\begin{aligned} K_{x,C} \supset \hat{\mathcal{O}}_{x,C} \dots \supset \wp_{x,C}^n \supset \dots, \\ K_C \supset \hat{\mathcal{O}}_C \dots \supset \wp_C^n \supset \wp_C^{n+1} \supset \dots, \\ K_x \supset \hat{\mathcal{O}}_x \dots \supset \wp_x^n \dots, \end{aligned}$$

and $\wp_x = \hat{\mathcal{O}}_x \cap \wp_{x,C}$.

Lemma 1 . We assume that the curve C is a locally complete intersection. Let $N_{X/C}$ be the normal sheaf for the curve C in X . For all $n \in \mathbb{Z}$ the maps

$$\prod_{x \in C} \wp_{x,C}^n \hat{\mathcal{F}}_{x,C} / \wp_{x,C}^{n+1} \hat{\mathcal{F}}_{x,C} \longrightarrow \mathbb{A}_{C,01}(\mathcal{F} \otimes \check{N}_{X/C}^{\otimes n}),$$

$$\prod_{x \in C} \wp_x^n \hat{\mathcal{F}}_x / \wp_x^{n+1} \hat{\mathcal{F}}_x \longrightarrow \mathbb{A}_{C,1}(\mathcal{F} \otimes \check{N}_{X/C}^{\otimes n}),$$

$$\wp_C^n \hat{\mathcal{F}}_C / \wp_C^{n+1} \hat{\mathcal{F}}_C \longrightarrow \mathbb{A}_{C,0}(\mathcal{F} \otimes \check{N}_{X/C}^{\otimes n}),$$

are bijective.

In general, we have an exact sequence

$$0 \longrightarrow \mathcal{J}^{n+1} \longrightarrow \mathcal{J}^n \longrightarrow \mathcal{J}^n|_C \longrightarrow 0$$

where $\mathcal{J} \subset \mathcal{O}_X$ is an ideal defining the curve C . In our case $\mathcal{J} = \mathcal{O}_X(-C)$ and $N_{X/C} = \mathcal{O}_X(C)|_C$. Thus the isomorphisms from the lemma are coming from the exact relative sequence

$$0 \longrightarrow \mathbb{A}_X(\mathcal{F}(-(n+1)C)) \longrightarrow \mathbb{A}_X(\mathcal{F}(-nC)) \longrightarrow \mathbb{A}_C(\mathcal{F}(-nC)|_C) \longrightarrow 0.$$

Lemma 2 . Let $P \in C$. For all $n \in \mathbb{Z}$ the complex

$$\wp_C^n \hat{\mathcal{F}}_C / \wp_C^{n+1} \hat{\mathcal{F}}_C \oplus \prod_{x \in C} \wp_x^n \hat{\mathcal{F}}_x / \wp_x^{n+1} \hat{\mathcal{F}}_x \longrightarrow \prod_{x \in C} \wp_{x,C}^n \hat{\mathcal{F}}_{x,C} / \wp_{x,C}^{n+1} \hat{\mathcal{F}}_{x,C}$$

is quasi-isomorphic to the complex

$$(A_C(\mathcal{F}) \cap \wp_C^n \hat{\mathcal{F}}_C) / (A_C(\mathcal{F}) \cap \wp_C^{n+1} \hat{\mathcal{F}}_C) \oplus \wp_P^n \hat{\mathcal{F}}_P / \wp_P^{n+1} \hat{\mathcal{F}}_P \longrightarrow \wp_{P,C}^n \hat{\mathcal{F}}_{P,C} / \wp_{P,C}^{n+1} \hat{\mathcal{F}}_{P,C}.$$

This lemma is an extension of the proposition 1 above. The proves of the both lemmas are straightforward and we will skip them.

Theorem 1 . Let X be a projective irreducible algebraic surface over a field k , $C \subset X$ be an irreducible projective curve, and $P \in C$ be a smooth point on both C and X . Let \mathcal{F} be a torsion free coherent sheaf on X .

Assume that the the surface $X - C$ is affine. Then the following complexes are quasi-isomorphic:

i) the adelic complex

$$\begin{aligned} \hat{\mathcal{F}}_\eta \oplus \prod_D \hat{\mathcal{F}}_D \oplus \prod_x \hat{\mathcal{F}}_x &\longrightarrow \prod_D (\hat{\mathcal{F}}_D \otimes K_D) \oplus \prod_x (\hat{\mathcal{F}}_x \otimes K_x) \oplus \prod_{x \in D} (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D}) \longrightarrow \\ &\longrightarrow \prod_{x \in D} (\hat{\mathcal{F}}_x \otimes K_{x,D}) \end{aligned}$$

for the sheaf \mathcal{F} and

ii) the complex

$$A(\mathcal{F}) \oplus A_C(\mathcal{F}) \oplus \hat{\mathcal{F}}_P \longrightarrow B_C(\mathcal{F}) \oplus B_P(\mathcal{F}) \oplus (\hat{\mathcal{F}}_P \otimes \hat{\mathcal{O}}_{P,C}) \longrightarrow \hat{\mathcal{F}}_P \otimes K_{P,C}$$

PROOF will be divided into several steps. We will subsequently transform the adelic complex checking that every time we get a quasi-isomorphic complex.

Step I. Consider the diagram

$$\begin{array}{ccccccc}
\prod_{D \neq C} \hat{\mathcal{F}}_D & \oplus & \prod_{x \in U} \hat{\mathcal{F}}_x & \longrightarrow & \prod_{D \neq C} \hat{\mathcal{F}}_D & \oplus & \prod_{x \in U} \hat{\mathcal{F}}_x & \oplus \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\hat{\mathcal{F}}_\eta & \oplus & \prod_D \hat{\mathcal{F}}_D & \oplus & \prod_x \hat{\mathcal{F}}_x & \longrightarrow & \prod_D K\hat{\mathcal{F}}_D & \oplus & \prod_x K\hat{\mathcal{F}}_x & \oplus \\
\parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\hat{\mathcal{F}}_\eta & \oplus & \hat{\mathcal{F}}_C & \oplus & \prod_{x \in C} \hat{\mathcal{F}}_x & \longrightarrow & (\prod_{D \neq C} K\hat{\mathcal{F}}_D / \hat{\mathcal{F}}_D \oplus K\hat{\mathcal{F}}_C) & \oplus & (\prod_{x \in U} K\hat{\mathcal{F}}_x / \hat{\mathcal{F}}_x \oplus \prod_{x \in C} K\hat{\mathcal{F}}_x) & \oplus \\
\oplus & \prod_{x \in D \neq C} \hat{\mathcal{F}}_{x,D} & \longrightarrow & \prod_{x \in D \neq C} \hat{\mathcal{F}}_{x,D} & & & & & & \\
\downarrow & & \downarrow & & & & & & & \\
\oplus & \prod_{x \in D} \hat{\mathcal{F}}_{x,D} & \longrightarrow & \prod_{x \in D} K\hat{\mathcal{F}}_{x,D} & & & & & & \\
\downarrow & & \downarrow & & & & & & & \\
\oplus & \prod_{x \in C} \hat{\mathcal{F}}_{x,C} & \longrightarrow & \prod_{x \in D \neq C} K\hat{\mathcal{F}}_{x,D} / \hat{\mathcal{F}}_{x,D} \oplus \prod_{x \in C} K\hat{\mathcal{F}}_{x,C} & & & & & &
\end{array}$$

where $U = X - C$. The middle row is the full adelic complex and the first row is an exact subcomplex. The commutativity of the upper squares is obvious. The exactness follows from the trivial

Lemma 3 . *Let $f_{1,2} : A_{1,2} \longrightarrow B$ be homomorphisms of abelian groups. The complex*

$$0 \longrightarrow A_1 \oplus A_2 \longrightarrow A_1 \oplus A_2 \oplus B \longrightarrow B \longrightarrow 0,$$

where $(a_1 \oplus a_2) \mapsto (a_1 \oplus -a_2 \oplus -f(a_1) + f(a_2))$, $(a_1 \oplus a_2 \oplus b) \mapsto (f(a_1) + f(a_2) + b)$, is exact.

The third row in the diagram is a quotient-complex by the subcomplex and we conclude that it is quasi-isomorphic to the adelic complex.

Step II. We can make the same step with the adelic complex for the sheaf \mathcal{F} on the surface U . By assumption the surface U is affine and we get an *exact* complex

$$\begin{aligned}
\hat{\mathcal{F}}_\eta / A &\longrightarrow \prod_{D \neq C} (\hat{\mathcal{F}}_D \otimes K_D) / \hat{\mathcal{F}}_D \oplus \prod_{x \in U} (\hat{\mathcal{F}}_x \otimes K_x) / \hat{\mathcal{F}}_x \longrightarrow \\
&\prod_{\substack{x \in U \\ x \in D \neq C}} (\hat{\mathcal{F}}_x \otimes K_{x,D}) / (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D}),
\end{aligned}$$

where $A = \Gamma(U, \mathcal{F})$.

Lemma 4 . *The complex*

$$0 \longrightarrow \prod_{x \in C} (\hat{\mathcal{F}}_x \otimes K_x) / B_x(\mathcal{F}) \longrightarrow \prod_{\substack{x \in C \\ x \in D \neq C}} (\hat{\mathcal{F}}_x \otimes K_{x,D}) / (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D}) \longrightarrow 0$$

is exact.

PROOF. The injectivity follows directly from the definition of the ring B_x . The surjectivity is the local strong approximation around the point $X \in X$ (see [13][§1],[6][ch.4]).

Step III. Take the sum of the two complexes from step II. Then we have a map of the complex we got in the step I to this complex

$$\begin{array}{ccccccc}
\hat{\mathcal{F}}_\eta & \oplus & \hat{\mathcal{F}}_C & \oplus & \prod_x \hat{\mathcal{F}}_x & \longrightarrow & (\prod_{D \neq C} K\hat{\mathcal{F}}_D/\hat{\mathcal{F}}_D \oplus K\hat{\mathcal{F}}_C) \oplus (\prod_{x \in U} K\hat{\mathcal{F}}_x/\hat{\mathcal{F}}_x \oplus \prod_{x \in C} K\hat{\mathcal{F}}_x) \oplus \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\hat{\mathcal{F}}_\eta/A & \oplus & (0) & \oplus & (0) & \longrightarrow & \prod_{D \neq C} K\hat{\mathcal{F}}_D/\hat{\mathcal{F}}_D \oplus (\prod_{x \in U} K\hat{\mathcal{F}}_x/\hat{\mathcal{F}}_x \oplus \prod_{x \in C} K\hat{\mathcal{F}}_x/B_x) \oplus \\
\oplus \prod_{x \in C} \hat{\mathcal{F}}_{x,C} & \longrightarrow & & & \prod_{x \in D \neq C} K\hat{\mathcal{F}}_{x,D}/\hat{\mathcal{F}}_{x,D} & \oplus & \prod_{x \in C} K\hat{\mathcal{F}}_{x,C} \\
\downarrow & & & & \downarrow & & \downarrow \\
\oplus (0) & \longrightarrow & \prod_{\substack{x \in U \\ x \in D \\ D \neq C}} K\hat{\mathcal{F}}_{x,D}/\hat{\mathcal{F}}_{x,D} \oplus \prod_{\substack{x \in C \\ x \in D \\ D \neq C}} K\hat{\mathcal{F}}_{x,D}/\hat{\mathcal{F}}_{x,D} & \oplus & (0) & &
\end{array}$$

For this map all the components which do not have arrows are mapped to zero. The diagram is commutative and the kernel of the map is equal to

$$A \oplus \hat{\mathcal{F}}_C \oplus \prod_{x \in C} \hat{\mathcal{F}}_x \longrightarrow K\hat{\mathcal{F}}_C \oplus \prod_{x \in C} B_x(\mathcal{F}) \oplus \prod_{x \in C} K\hat{\mathcal{F}}_x \longrightarrow \prod_{x \in C} K\hat{\mathcal{F}}_{x,C}.$$

We conclude that this complex is quasi-isomorphic to the adelic complex.

Step IV. Using the embedding $\hat{\mathcal{F}}_x \longrightarrow B_x(\mathcal{F})$ and lemma 3 we have an exact complex and it's embedding into the complex of the step III:

$$\begin{array}{ccccccc}
\prod_{x \in C-P} \hat{\mathcal{F}}_x & \longrightarrow & \prod_{x \in C-P} B_x(\mathcal{F}) & \oplus & \prod_{x \in C-P} \hat{\mathcal{F}}_x & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
A \oplus \hat{\mathcal{F}}_C \oplus \prod_{x \in C} \hat{\mathcal{F}}_x & \longrightarrow & K\hat{\mathcal{F}}_C \oplus \prod_{x \in C} B_x(\mathcal{F}) & \oplus & \prod_{x \in C} \hat{\mathcal{F}}_{x,C} & \longrightarrow & \\
& & \longrightarrow \prod_{x \in C-P} B_x(\mathcal{F}) & & & & \\
& & \downarrow & & & & \\
& & \longrightarrow \prod_{x \in C} K\hat{\mathcal{F}}_{x,C} & & & &
\end{array}$$

As a result we get the factor-complex

$$\begin{aligned}
A \oplus \hat{\mathcal{F}}_C \oplus \hat{\mathcal{F}}_P &\longrightarrow K\hat{\mathcal{F}}_C \oplus B_P(\mathcal{F}) \oplus \prod_{x \in C-P} \hat{\mathcal{F}}_{x,C}/\hat{\mathcal{F}}_x \oplus \hat{\mathcal{F}}_{P,C} \longrightarrow \\
&\longrightarrow \prod_{x \in C-P} K\hat{\mathcal{F}}_{x,C}/B_x(\mathcal{F}) \oplus K\hat{\mathcal{F}}_{P,C}.
\end{aligned}$$

Step V. Now we need

Lemma 5 *The complex*

$$0 \longrightarrow (\hat{\mathcal{F}}_C \otimes K_C)/B_C(\mathcal{F}) \longrightarrow \prod_{x \in C-P} (\hat{\mathcal{F}}_x \otimes K_{x,C})/B_x(\mathcal{F}) \longrightarrow 0$$

is exact.

PROOF. The injectivity is again the definition of the B_C and the surjectivity follows from the strong approximation on the curve C ([5]) and lemma 2 above.

As a corollary we have an isomorphism

$$\hat{\mathcal{F}}_C/A_C(\mathcal{F}) \xrightarrow{\cong} \prod_{x \in C-P} (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,C})/\hat{\mathcal{F}}_x,$$

where

$$A_C(\mathcal{F}) := B_C(\mathcal{F}) \cap \hat{\mathcal{F}}_C.$$

Combining the isomorphisms from the lemma and its corollary into a single complex of length 2, we get the diagram

$$\begin{array}{ccccccc} A \oplus \hat{\mathcal{F}}_C \oplus \hat{\mathcal{F}}_P & \longrightarrow & K\hat{\mathcal{F}}_C \oplus B_P(\mathcal{F}) \oplus (\prod_{x \in C-P} \hat{\mathcal{F}}_{x,C}/\hat{\mathcal{F}}_x \oplus \hat{\mathcal{F}}_{P,C}) & \longrightarrow \\ \downarrow & & \downarrow & \\ (0) \oplus \hat{\mathcal{F}}_C/A_C \oplus (0) & \longrightarrow & K\hat{\mathcal{F}}_C/B_C \oplus (0) \oplus \prod_{x \in C-P} \hat{\mathcal{F}}_{x,C}/\hat{\mathcal{F}}_x & \longrightarrow \\ & \longrightarrow & \prod_{x \in C-P} K\hat{\mathcal{F}}_{x,C}/B_x(\mathcal{F}) \oplus K\hat{\mathcal{F}}_{P,C} & \\ & & \downarrow & \\ & \longrightarrow & \prod_{x \in C-P} K\hat{\mathcal{F}}_{x,C}/B_x(\mathcal{F}) \oplus (0) & \end{array}$$

The kernel of the map of the complexes is obviously equal to

$$A(\mathcal{F}) \oplus A_C(\mathcal{F}) \oplus \hat{\mathcal{F}}_P \longrightarrow B_C(\mathcal{F}) \oplus B_P(\mathcal{F}) \oplus (\hat{\mathcal{F}}_P \otimes \hat{\mathcal{O}}_{P,C}) \longrightarrow (\hat{\mathcal{F}}_P \otimes K_{P,C})$$

and we arrive to the conclusion of the theorem.

REMARK 1. Sometimes we will call the complex from the theorem as the *restricted adelic complex*.

Lemma 6 .Let X be a projective irreducible variety over a field k and $\mathcal{O}(1)$ be a very ample sheaf on X . Then

1. The following conditions are equivalent

- i) X is a Cohen-Macaulay variety
- ii) for any locally free sheaf \mathcal{F} on X and $i < \dim(X)$ $H^i(X, \mathcal{F}(n)) = (0)$ for $n \ll 0$

2. If X is normal of dimension > 1 then for any locally free sheaf \mathcal{F} on X $H^1(X, \mathcal{F}(n)) = (0)$ for $n \ll 0$

PROOF see in [8][ch. III, Thm. 7.6, Cor. 7.8]. We only note that the last statement is known as the lemma of Enriques-Severi-Zariski. For dimension 2 every normal variety is Cohen-Macaulay and thus the second claim follows from the first one.

Proposition 2 . *Let \mathcal{F} be a locally free coherent sheaf on the projective irreducible surface X .*

Assume that the local rings of the X are Cohen-Macaulay and the curve C is a locally complete intersection. Then, inside the field $K_{P,C}$, we have

$$B_C(\mathcal{F}) \cap B_P(\mathcal{F}) = A(\mathcal{F}).$$

PROOF will be done in several steps.

STEP 1. If we know the proposition for a sheaf \mathcal{F} then it is true for the sheaf $\mathcal{F}(nC)$ for any $n \in \mathbb{Z}$. Thus taking a twist by $\mathcal{O}(n)$ we can assume that $\deg_C(\mathcal{F}) < 0$.

STEP 2. Now we show that $A_C(\mathcal{F}) \cap \hat{\mathcal{F}}_P = (0)$. The filtrations from lemma 1 gives the corresponding filtration of the group $A_C(\mathcal{F})$. Lemma 2 implies that

$$\frac{(A_C(\mathcal{F}) \cap \hat{\mathcal{F}}_P) \cap \wp^n \hat{\mathcal{F}}_P}{(A_C(\mathcal{F}) \cap \hat{\mathcal{F}}_P) \cap \wp^{n+1} \hat{\mathcal{F}}_P} \cong \Gamma(C, \mathcal{F} \otimes \check{N}_{X/C}^{\otimes n}).$$

Since $\deg_C(\mathcal{F}) < 0$, $N_{X/C} = \mathcal{O}_X(C)|_C$ and $\deg_C(N_{X/C}) > 0$ we get that the last group is trivial.

STEP 3. The next step is to prove the equality:

$$B_C(\mathcal{F}(-D)) \cap B_P(\mathcal{F}(-D)) = A(\mathcal{F}(-D)),$$

where D is an sufficiently ample divisor on X distinct from the curve C . By theorem 1 the cohomology of $\mathcal{F}_X(-D)$ can be computed from the complex

$$\begin{aligned} A(\mathcal{F}(-D)) \oplus A_C(\mathcal{F}(-D)) \oplus \hat{\mathcal{F}}_P(-D) &\longrightarrow B_C(\mathcal{F}(-D)) \oplus B_P(\mathcal{F}(-D)) \oplus \hat{\mathcal{F}}_{P,C}(-D) \\ &\longrightarrow K\mathcal{F}_{P,C}. \end{aligned}$$

Now take $a_{01} \in B_C(\mathcal{F}(-D))$, $a_{02} \in B_P(\mathcal{F}(-D))$ such that $a_{01} + a_{02} = 0$. They define an element $(a_{01} \oplus a_{02} \oplus 0)$ in the middle component of the complex. By our condition for D and the lemma 6 we have $H^1(X, \mathcal{F}_X(-D)) = (0)$ and thus there exist $a_0 \in A(\mathcal{F}(-D))$, $a_1 \in A_C(\mathcal{F}(-D))$, $a_2 \in \hat{\mathcal{F}}_P(-D)$ such that $a_{01} = a_0 - a_1$, $a_{02} = a_2 - a_0$, $0 = a_1 - a_2$.

By the second step $a_1 = a_2 \in (A_C(\mathcal{F}(-D)) \cap \hat{\mathcal{F}}_P(-D)) \subset A_C(\mathcal{F}) \cap \hat{\mathcal{F}}_P = (0)$ and, consequently, we have $a_{01} (= -a_{02}) \in A(\mathcal{F}(-D))$.

STEP 4. The last step is to take two distinct divisors D, D' such that $D \cap D' \subset C$. Since C is a hyperplane section we can choose for D, D' two hyperplane sections whose intersection belongs to C . Therefore their ideals in the ring $A(\mathcal{F})$ are relatively prime and

$$A(\mathcal{F}) = A(\mathcal{F}(-D)) + A(\mathcal{F}(-D')) \ni 1 = a + a', a \in A(\mathcal{F}(-D)), a' \in A(\mathcal{F}(-D')).$$

If now $b \in B_C(\mathcal{F}) \cap B_P(\mathcal{F})$, then $b = ba + ba'$, where $ba \in B_C(\mathcal{F}(-D)) \cap B_P(\mathcal{F}(-D))$, $ba' \in B_C(\mathcal{F}(-D')) \cap B_P(\mathcal{F}(-D'))$. We see that $b \in A(\mathcal{F})$ by the previous step.

REMARK 2. The method we have used cannot be applied if our variety is not Cohen-Macaulay (by lemma 6 above). It would be interesting to know how to extend the result to the arbitrary surfaces X and the sheaves \mathcal{F} such that \mathcal{F} are locally free outside C . The last condition is really necessary.

REMARK 3. This proposition is a version for the reduced adelic complex of the corresponding result for the full complex. Namely, $\mathbb{A}_{X,01} \cap \mathbb{A}_{X,02} = \mathbb{A}_{X,0}$, see [6, ch.IV]. This should be generalized to arbitrary dimension n in the following way.

Let $I, J \subset [0, 1, \dots, n]$ and

$$\mathbb{A}_{X,I}(\mathcal{F}) = \left(\prod_{\{\text{codim}\eta_0, \text{codim}\eta_1, \dots\} \in I} K_{\eta_0, \eta_1, \dots} \right) \otimes \mathcal{F}_{\eta_0} \bigcap \mathbb{A}_X(\mathcal{F}).$$

Then we have

$$\mathbb{A}_{X,I}(\mathcal{F}) \cap \mathbb{A}_{X,J}(\mathcal{F}) = \mathbb{A}_{X,I \cap J}(\mathcal{F})$$

for a locally free \mathcal{F} and a Cohen-Macaulay X .

EXAMPLE. Let $X = \mathbf{P}_2 \supset C = \mathbf{P}_1 \ni P$. We introduce homogenous coordinates $(x_0 : x_1 : x_2)$ such that $C = (x_0 = 0)$; $P = (x_0 = x_1 = 0)$ and $U = X - C = \text{Spec} k[x, y]$ with $x = x_1/x_0, y = x_2/x_0$. Then $k(C) = k(y/x), x^{-1}$ is the last parameter for any two-dimensional local field $K_{Q,C}$ with $Q \neq P$. For local field $K_{P,C}$ we have

$$K_{P,C} = k((u))((t)), u = xy^{-1}, t = y^{-1}.$$

Then we can easily compute all the rings from the complex of theorem 1 for the sheaf \mathcal{O}_X .

$$\begin{aligned} B_P &= k[[u]]((t)) \\ B_C &= k[u^{-1}]((u^{-1}t)) \\ \hat{\mathcal{O}}_{P,C} &= k((u))[[t]] \\ A = \Gamma(U, \mathcal{O}_X) &= k[ut^{-1}, t^{-1}] \\ A_C &= k[u^{-1}][[u^{-1}t]] \\ \hat{\mathcal{O}}_P &= k[[u, t]] \end{aligned}$$

We can draw the subspaces as some subsets of the plane according to the supports of the elements of the subspaces (on the plane with coordinates (i, j) for elements $u^i t^j \in K_{P,C}$). Then the first three subspaces $B_P, B_C, \hat{\mathcal{O}}_{P,C}$ will correspond to some halfplanes and the subspaces $A, A_C, \hat{\mathcal{O}}_P$ to the intersections of them.

2 Main Theorem

We need the following well known result.

Lemma 7 . *Let X be an projective variety, \mathcal{F} be a coherent sheaf on X and C be an ample divisor on X . If*

$$S = \oplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nC)), \quad F = \oplus_{n \geq 0} \Gamma(X, \mathcal{F}(nC)),$$

then

$$X \cong \text{Proj}(S), \quad \mathcal{F} \cong \text{Proj}(F).$$

PROOF. Let mC be a very ample divisor, $S = \oplus_{n \geq 0} S_n$ and $S' := \oplus_{n \geq 0} S_{nm}$. Then by [7][prop. 2.4.7]

$$\text{Proj}(S') \cong \text{Proj}(S).$$

The divisor mC defines an embedding $i : X \longrightarrow \mathbf{P}$ to a projective space such that $i^* \mathcal{O}_{\mathbf{P}}(1) = \mathcal{O}_X(mC)$. Let $\mathcal{J}_X \subset \mathcal{O}_{\mathbf{P}}$ be an ideal defined by X . If

$$I := \oplus_{n \geq 0} \Gamma(\mathbf{P}, \mathcal{J}_X(n)), \\ A := \oplus_{n \geq 0} \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(n)),$$

then $I \subset A$ and by [7][prop. 2.9.2]

$$\text{Proj}(A/I) \cong X.$$

We have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{J}_X(n) \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{O}_X(n) \longrightarrow 0,$$

which implies the sequence

$$0 \longrightarrow \bigoplus_{n \geq 0} \Gamma(\mathcal{J}_X(n)) \longrightarrow \bigoplus_{n \geq 0} \Gamma(\mathcal{O}_{\mathbf{P}}(n)) \longrightarrow \bigoplus_{n \geq 0} \Gamma(\mathbf{P}, \mathcal{O}_X(n)) \longrightarrow \bigoplus_{n \geq 0} H^1(\mathbf{P}, \mathcal{J}_X(n)).$$

Here the last term is trivial for sufficiently large n . The first three terms are equal correspondingly to I, A and S' . It means that the homogenous components of A/I and $S' \supset A/I$ are equal for sufficiently big degrees.

By [7][prop. 2.9.1]

$$\text{Proj}(A/I) \cong \text{Proj}(S'),$$

and combining everything together we get the statement of the lemma. The statement concerning the sheaf \mathcal{F} can be proved along the same line.

Let us first explain the Krichever correspondence for dimension 1.

DEFINITION 2.

\mathcal{M}_1	$:= \{C, P, z, \mathcal{F}, e_P\}$
C	projective irreducible curve $/k$
$P \in C$	a smooth point
z	formal local parameter at P
\mathcal{F}	torsion free rank r sheaf on C
e_P	a trivialization of \mathcal{F} at P

Independently, we have the field $K = k((z))$ of Laurent power series with filtration $K(n) = z^n k[[z]]$. Let $K_1 := K(0)$. If $V = k((z))^{\oplus r}$ then $V(n) = K(n)^{\oplus r}$ and $V_1 := V(0)$.

Theorem 2 [11]. *There exists a canonical map*

$$\Phi_1 : \mathcal{M}_1 \longrightarrow \{\text{vector subspaces } A \subset K, W \subset V\}$$

such that

i) the cohomology of complexes

$$A \oplus K_1 \longrightarrow K, \quad W \oplus V_1 \longrightarrow V$$

are isomorphic to $H^(C, \mathcal{O}_C)$ and $H^*(C, \mathcal{F})$, respectively*

ii) if $(A, W) \in \text{Im } \Phi_1$ then $A \cdot A \subset A, A \cdot W \subset W$,

iii) if $m, m' \in \mathcal{M}_1$ and $\Phi_1(m) = \Phi_1(m')$ then m is isomorphic to m'

PROOF. If $m = (C, P, z, \mathcal{F}, e_P) \in \mathcal{M}_1$ then we put

$$A := \Gamma(C - P, \mathcal{O}_C),$$

$$W := \Gamma(C - P, \mathcal{F}).$$

Also we have

$$\hat{\mathcal{O}}_P = k[[z]], \quad K_P = k((z)),$$

$$\mathcal{F}_P = \mathcal{O}_{Pe_P} = \mathcal{O}_P^{\oplus r}, \quad \hat{\mathcal{F}}_P = \hat{\mathcal{O}}_P^{\oplus r}.$$

This defines the point $\Phi_1(m) \in \mathcal{M}_1$. Indeed, for the subspace W we have the following canonical identifications

$$\Gamma(C - P, \mathcal{F}) \subset \mathcal{F}_\eta \otimes_{\mathcal{O}_P} K_P = \hat{\mathcal{F}}_P \otimes K_P = \hat{\mathcal{O}}_P^{\oplus r} \otimes K_P = k((z))^{\oplus r}.$$

The same works for the subspace A .

The property ii) is obvious, the property i) follows from the proposition 1. To get iii) let us start with a point $\Phi_1(m) = (A, W)$. The standard valuation on K gives us increasing filtrations $A(n) = A \cap K(n)$ and $W(n) = W \cap V(n)$ on the spaces A and W . Then we have

$$\begin{aligned} C - P &= \text{Spec}(A), \\ C &= \text{Proj}(\oplus_n A(n)), \\ \mathcal{F} &= \text{Proj}(\oplus_n W(n)), \end{aligned}$$

by lemma 6. Thus we can reconstruct the quintuple m from the point $\Phi_1(m)$.

REMARK 4. It is possible to replace the ground field k in the Krichever construction by an arbitrary scheme S , see [17].

Now we move to the case of algebraic surfaces. The corresponding data has the following

DEFINITION 3.

\mathcal{M}_2	$:= \{X, C, P, (z_1, z_2), \mathcal{F}, e_P\}$
X	projective irreducible surface $/k$
$C \subset X$	projective irreducible curve $/k$
$P \in C$	a smooth point on X and C
z_1, z_2	formal local parameter at P such that $(z_2 = 0) = C$ near P
\mathcal{F}	torsion free rank r sheaf on X
e_P	a trivialization of \mathcal{F} at P

Then we have

$$\hat{\mathcal{O}}_{X,P} = k[[z_1, z_2]], \quad K_{P,C} = k((z_1))((z_2)),$$

$$\hat{\mathcal{F}}_P = \hat{\mathcal{O}}_P e_P = \hat{\mathcal{O}}_P^{\oplus r}.$$

For the field $K = k((z_1))((z_2))$ we have the following filtrations and subspaces:

$$\begin{aligned} K_{02} &= k[[z_1]]((z_2)), \\ K_{12} &= k((z_1))[[z_2]], \\ K(n) &= z_2^n K_{12}. \end{aligned}$$

Taking the direct sums we introduce the subspaces $V_{02}, V_{12}, V(n)$ of the space $V = K^{\oplus r}$.

Theorem 3 . *Let C be a hyperplane section on the surface X . Then there exists a canonical map*

$$\Phi_2 : \mathcal{M}_2 \longrightarrow \{\text{vector subspaces } B \subset K, W \subset V\}$$

such that

i) for all n the complexes

$$\frac{B \cap K(n)}{B \cap K(n+1)} \oplus \frac{K_{02} \cap K(n)}{K_{02} \cap K(n+1)} \longrightarrow \frac{K(n)}{K(n+1)}$$

$$\frac{W \cap V(n)}{W \cap V(n+1)} \oplus \frac{V_{02} \cap V(n)}{V_{02} \cap V(n+1)} \longrightarrow \frac{V(n)}{V(n+1)}$$

are Fredholm of index $\chi(C, \mathcal{O}_C) + nC.C$ and $\chi(C, \mathcal{F}|_C) + nC.C$, respectively

ii) the cohomology of complexes

$$(B \cap K_{02}) \oplus (B \cap K_{12}) \oplus (K_{02} \cap K_{12}) \longrightarrow B \oplus K_{02} \oplus K_{12} \longrightarrow K$$

$$(W \cap V_{02}) \oplus (W \cap V_{12}) \oplus (V_{02} \cap V_{12}) \longrightarrow W \oplus V_{02} \oplus V_{12} \longrightarrow V$$

are isomorphic to $H^*(X, \mathcal{O}_X)$ and $H^*(X, \mathcal{F})$, respectively

iii) if $(B, W) \in \text{Im } \Phi_2$ then $B \cdot B \subset B, B \cdot W \subset W$

iv) for all n the map

$$\begin{aligned} & (C, P, z_1|_C, \mathcal{F}(nC)|_C, e_P(n)|_C) \mapsto \\ & \mapsto \left(\frac{B \cap K(n)}{B \cap K(n+1)} \subset \frac{K(n)}{K(n+1)} = k((z_1)), \right. \\ & \left. \frac{W \cap V(n)}{W \cap V(n+1)} \subset \frac{V(n)}{V(n+1)} = k((z_1))^{\oplus r} \right) \end{aligned}$$

coincides with the map Φ_1 .

v) let the sheaf \mathcal{F} be locally free and the surface X be Cohen-Macaulay. If $m, m' \in \mathcal{M}_1$ and $\Phi_2(m) = \Phi_2(m')$ then m is isomorphic to m'

PROOF. If $m = (X, C, P, (z_1, z_2), \mathcal{F}, e_P) \in \mathcal{M}_2$ then to define the map Φ_2 we put

$$\begin{aligned} B &= B_C(\mathcal{O}_X), \\ W &= B_C(\mathcal{F}), \\ \Phi_2(m) &= (B, W). \end{aligned}$$

Since we have the local coordinates $z_{1,2}$ and the trivialization e_P the subspaces B and W will belong to the space $k((z_1))((z_2))$ exactly as in the case of dimension 1 considered above.

We note that our condition on the curve C implies that C is a Cartier divisor and the surface $X - C$ is affine.

The property i) follows from lemma 2, the property ii) follows from theorem 1. The property iii) is trivial again, to get iv) one needs again to apply lemma 2 and to get v) it is enough to use proposition 2 and lemma 7. They show that given a point $(B, W) \in \mathcal{M}_2$ such that $(B, W) = \Phi_2(m)$ we can reconstruct the data m up to an isomorphism.

REMARK 5. The property v) of the theorem cannot be extended to the arbitrary torsion free sheaves on X . We certainly cannot reconstruct such sheaf if it is not locally free outside C . Indeed, if $\mathcal{F}, \mathcal{F}'$ are two sheaves and there is a monomorphism $\mathcal{F}' \longrightarrow \mathcal{F}$ such that \mathcal{F}/\mathcal{F}' has support in $X - C$ then the restricted adelic complexes for the sheaves $\mathcal{F}, \mathcal{F}'$ are isomorphic.

REMARK 6. A definition of the map Φ_n for all n was suggested in [12]. It has the properties that correspond to the properties i) - v) of the theorem.

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